

# Black hole as emergent holographic geometry of weakly interacting hot Yang-Mills gas\*

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**ABSTRACT:** We demonstrate five-dimensional anti-de Sitter black hole emerges as dual geometry holographic to weakly interacting  $\mathcal{N} = 4$  superconformal Yang-Mills theory. We first note that an ideal probe of the dual geometry is the Yang-Mills instanton, probing point by point in spacetime. We then study instanton moduli space at finite temperature by adopting Hitchin's proposal that geometry of the moduli space is definable by Fisher-Rao "information geometry". In Yang-Mills theory, the information metric is measured by a novel class of gauge-invariant, nonlocal operators in the instanton sector. We show that the moduli space metric exhibits (1) asymptotically anti-de Sitter, (2) horizon at radial distance set by the Yang-Mills temperature, and (3) after Wick rotation of the moduli space to the Lorentzian signature, a singularity at the origin. We argue that the dual geometry emerges even for rank of gauge groups of order unity and for weak 't Hooft coupling.

**KEYWORDS:** AdS-CFT Correspondence, Statistical Methods, Black Holes in String Theory.

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## 1. Background

Maldacena’s gauge-gravity correspondence [1] refers to the remarkable relation between 4-dimensional conformal gauge theory and 5-dimensional string theory on anti-de Sitter space. The gauge theory is characterized by gauge group  $SU(N)$  and ‘t Hooft coupling  $\lambda^2 = g_{YM}^2 N$ , while string theory is characterized by coupling  $g_{st}$  and curvature scale (in string scale unit)  $R$  of the 5-dimensional anti-de Sitter space, whose metric in Poincaré coordinates is given by

$$ds_0^2 = R^2 \left( \frac{1}{u^2} [-dt^2 + d\mathbf{x}^2] + \frac{du^2}{u^2} \right). \tag{1.1}$$

The correspondence is valid in the limit the string theory is weakly coupled and anti-de Sitter space is nearly flat, and identifies coupling parameters on each side as

$$N \sim \frac{1}{g_{st}} \gg 1 \quad \text{and} \quad \lambda^2 \sim R^4 \gg 1. \tag{1.2}$$

The correspondence is a new kind of strong-weak coupling duality, and has been established firmly in the limit of (1.2).

Maldacena’s correspondence is extendible to Yang-Mills theory at finite temperature. The dual string theory is then defined on Schwarzschild black hole in 5-dimensional anti-de Sitter space:

$$ds_T^2 = R^2 \left[ -\frac{1}{u^2} \left( 1 - \frac{u^4}{u_0^4} \right) dt^2 + \left( 1 - \frac{u^4}{u_0^4} \right)^{-1} \frac{du^2}{u^2} + \frac{1}{u^2} d\mathbf{x}^2 \right], \tag{1.3}$$

whose surface gravity at the horizon  $u = u_0$  set by the Yang-Mills temperature  $T$ . The Maldacena’s correspondence, as is formulated, is valid for large  $N$  and strong ‘t Hooft coupling, see (2). On the other hand, enormous insight on gauge and string theories would be gained by understanding other regime of the coupling parameters. So, we consider taking extreme opposite regime

$$N \sim \mathcal{O}(1) \quad \text{and} \quad \lambda \ll 1, \tag{1.4}$$

and pose the following questions. Despite being in the regime of strongly interacting string theory, can a black hole geometry (3) or close to it be reconstructed out of weakly interacting Yang-Mills gas? Can we explore interior of the horizon and see if the black hole singularity resolved by strong stringy and quantum effects? If resolved, what are universal features of the resolved geometry? Undoubtedly, affirmative answers to these questions would yield far reaching consequences to our ultimate understanding on gauge theory, quantum gravity, string theory, black hole, and (spacelike) singularity.

These questions are interesting, but one may subscribe a certain doubt because eq. (1.4) lies outside the validity range of Maldacena’s correspondence. On the other hand, there are physical quantities whose behavior seems to support smooth interpolation between weak and strong coupling regimes. Take the thermodynamic free energy density  $\mathcal{F}(N, \lambda)$  of hot Yang-Mills gas. At large  $N$  but at arbitrary  $\lambda$ , the free energy density behaves as  $\mathcal{F}(N, T) = h(\lambda) (-\frac{1}{6}\pi^2 N^2 T^4)$ . At strong ‘t Hooft coupling regime,  $\lambda \gg 1$ , the leading-order correction was computed from string worldsheet corrections to the Type IIB supergravity and hence to the black hole geometry. It shows that  $h(\lambda)$  increases monotonically from 3/4 as  $\lambda$  is decreased [2]. At weak ‘t Hooft coupling regime,  $\lambda \ll 1$ , the leading-order corrections was computed up to two loop by thermal perturbation theory for Yang-Mills theory, taking into account of bubble resummation and electric screening thereof. With Padé approximation, it was shown that  $h(\lambda)$  decreases monotonically from 1 as  $\lambda$  is increased [3].

In this work, we shall study  $\mathcal{N} = 4$  super Yang-Mills theory at finite temperature at weak coupling and look for a signature, if any, of holography [4] and of emergent black hole geometry. A moment of thought indicates that perturbative gluon dynamics at weak coupling is an unlikely setup to discover such geometry, and, by S-duality, semiclassical solitons (such as instantons or monopoles) at strong coupling limit is likewise an unlikely setup. We will therefore focus on instantons in weak coupling limit, where semiclassical treatment is well justified. Morally speaking, this is S-dual to gluon dynamics at strong coupling limit — the situation where Maldacena’s correspondence was extensively confirmed. Moreover, instanton in Yang-Mills theory is the counterpart of D-instanton in Type IIB string theory on 5d anti-de Sitter space. The D-instanton is an ideal probe since it has sub-string scale size at weak coupling and it probes the spacetime point by point. Built upon a prescription by Hitchin [5], we will show that a kind of black hole geometry in asymptotic anti-de Sitter background emerges as the ”information geometry” [6, 7] for the moduli space of Yang-Mills instanton at finite temperature.

## 2. Theoretical procedure

The Yang-Mills instanton for gauge group  $G = \text{SU}(2)$  is a self-dual configuration of the field strength  $F_{mn}^a$ :  $F_{mn}^a = \frac{1}{2}\epsilon_{mnpq} F_{pq}^a$ . For a single instanton, the Yang-Mills field strength and Lagrangian is given by

$$F_{mn}^a = -\eta_{mn}^a \frac{\rho^2}{[(x-a)^2 + \rho^2]^2}, \quad \text{and} \quad \mathcal{L}_{\text{YM}} \equiv \frac{1}{2} \text{Tr} F_{mn}^2 = \frac{\rho^4}{[(x-a)^2 + \rho^2]^4}. \quad (2.1)$$

Notice that  $\mathcal{L}_{\text{YM}}$  is a function of both four-dimensional coordinates  $x^m$  on  $\mathbf{R}^4$  and five-dimensional coordinates  $Z^A \equiv (a^m, \rho)$  (which parametrize center and size of the instanton) on the instanton moduli space  $\mathcal{M}$  whose topology is  $\mathbf{R}_4 \times \mathbf{R}_+$ .

The geometry of  $\mathcal{M}$  is definable by an appropriate choice of the metric. A familiar choice, studied extensively, is the ‘ $L^2$ -metric’  $G_{\text{AB}}^{L^2} := \int_{\mathbb{R}^4} g^{mn}(x) (\partial_A A_m \partial_B A_n)(x; Z)$ . It yields, for the base manifold  $\mathbf{R}^4$ ,  $ds^2(G^{L^2}) = d\rho^2 + da^2 = dZ^2$ . As pointed out in [5], the  $L^2$ -metric is not convenient for studying *differential geometry* of the moduli space  $\mathcal{M}$ , since the metric does not display the underlying conformal invariance and the moduli space is geometrically incomplete as the small instanton singularity  $\rho = 0$  is at a finite distance from a generic point on  $\mathcal{M}$ . Also, the metric depends sensitively on topology and choice of the metric on the base manifold  $B$ , and for  $G = \text{SU}(N)$ , one instanton moduli space is  $4N$ -dimensional, depending on the rigid  $\text{SU}(N)$  gauge transformations. In the context of Maldacena’s gauge-gravity correspondence, it is worthwhile to recall that the ADHM instanton for  $\mathcal{N} = 4$  super Yang-Mills theory [8] yielded  $AdS_5 \times S_5$  *after* integrating out the zero-modes for  $\text{U}(N)$  global gauge rotations by the large  $N$  saddle point approximation. As a result, the computation was reduced entirely to  $G = \text{SU}(2)$  instanton calculus.

To keep the underlying symmetry manifest and better aid differential geometric properties, Hitchin [5] proposed an alternative definition of the moduli space geometry. For the set of instantons with a fixed instanton number  $Q$ , the Lagrangian density  $\mathcal{L}_{\text{YM}}$  may be considered as a probability distribution functions:  $\int_{\mathbb{R}^4} \mathcal{L}_{\text{YM}} = Q$ . Hitchin then proposes to utilize so-called Fisher-Rao’s information metric to describe the geometry of the instanton moduli space  $\mathcal{M}$ . Extending his prescription, we propose the information metric in gauge theory in terms of quantum average of nonlocal, gauge invariant operator:

$$G_{\text{AB}}^{\text{info}}(Z) \equiv \int_{\mathbb{R}^4} \langle \mathcal{L}_{\text{YM}} (\partial_A \log \mathcal{L}_{\text{YM}}) (\partial_B \log \mathcal{L}_{\text{YM}}) \rangle. \tag{2.2}$$

Here, the bracket abbreviates normalized functional integral over the gauge field configurations (whose definition involves the standard  $L^2$ -metric for the functional integral measure). At leading order in weak coupling regime, one simply evaluates (2.2) for the instanton solution, obtaining Hitchin’s original prescription.

The information metric (2.2) probes variation of the instanton density over the moduli space  $\mathcal{M}$ . Indeed, from (2.1), information metric of the unit charge,  $\text{SU}(2)$  Yang-Mills instanton is readily extracted as

$$ds^2(G^{\text{info}}) = c \frac{d\rho^2 + da^2}{\rho^2}. \tag{2.3}$$

So, identifying  $\rho, a^m$  with  $u, x^m$  in (2), we constructed 5-dimensional Euclidean anti-de Sitter space  $\mathbf{H}^5$  as the emergent geometry of weakly coupled Yang-Mills theory. Notice that the information metric (2.3) exhibits underlying conformal invariance manifestly, geodesic completeness (small instanton singularity is at infinite distance from a generic point in the interior), and invariance under the global gauge rotation.

The above result may be interpreted as follows. In Maldacena’s gauge-gravity correspondence, classical equation of motion of dilaton  $\Phi$ , as derived by the variation of the

supergravity effective action  $W_{\text{sugra}}$ , is sourced by the Lagrangian density of the  $\mathcal{N} = 4$  super Yang-Mills theory [9, 10]. Thus, for an instanton configuration of a fixed topological charge,

$$\left(\frac{\delta W_{\text{sugra}}}{\delta \Phi}\right)_{\text{D-instanton}} = \frac{1}{g_{\text{YM}}^2} \langle \mathcal{L} \rangle_{\text{instanton}}. \tag{2.4}$$

As seen above, the right-hand side depends not only on the coordinates of the base manifold  $B = \mathbf{R}^4$  but also on the coordinates on  $\mathcal{M}$ . Therefore, the Lagrangian density for the instanton, equivalently, topological charge density is interpretable as the bulk-to-boundary propagator for a massless spin-0 field. In fact, one can show that

$$\square_{\mathbb{H}_5} \mathcal{L}_{\text{YM}}(x, Z) = 0 \quad \text{and} \quad \lim_{\rho \rightarrow 0} \mathcal{L}_{\text{YM}}(x, Z) = \delta^{(4)}(x - x_0),$$

both of which are precisely the defining equations for the bulk-boundary propagators in AdS space. One can further construct bulk-boundary propagators for massive spin-0 fields with  $m^2 = \Delta(\Delta - 4)$  by raising power of the instanton charge density  $\mathcal{L}_{\text{YM}}$ :

$$\square_{\mathbb{H}_5} \left(\mathcal{L}_{\text{YM}}(x, Z)\right)^{\Delta/4} = \Delta(\Delta - 4) \left(\mathcal{L}_{\text{YM}}(x, Z)\right)^{\Delta/4}. \tag{2.5}$$

The rate of variation of the instanton density over the moduli space  $\mathcal{M}$  is measurable by the logarithmic derivative  $v_A(x; Z) \equiv \partial_A \log \mathcal{L}_{\text{YM}}(x : Z)$ . By explicit computation [11], one can show that these logarithmic vector fields obey the ‘Hamilton-Jacobi’ equation  $\|v\|_{\text{info}}^2 \equiv G_{\text{info}}^{\text{AB}} v_A v_B = 1$  for *every* point  $x^m$  on the base manifold  $B = \mathbf{R}^4$ . Therefore,  $v_A$  may be interpreted as the velocity field, and  $G_{\text{AB}}^{\text{info}}$  a viable metric defining the geodesic flows on the moduli space  $\mathcal{M}$ .

Salient features of the information metric are [11]

- Information metric  $G_{\text{AB}}$  is Einstein. Furthermore, perturbation of information metric  $G_{\text{AB}}$  is the bulk-boundary propagator for ‘graviton’.
- The instanton charge density  $\mathcal{L}_{\text{YM}}$  is the bulk-boundary propagator for massless scalar fields. This relation holds also for perturbation of  $\mathcal{L}_{\text{YM}}$ .
- The geodesic distance, once properly regularized, between the boundary point  $x^m$  of the base manifold  $B = \mathbf{R}^4$  and the bulk point  $Z^A$  is given by  $\log \mathcal{L}(x : Z)$ .

We shall now study the information geometry of instanton moduli space for Yang-Mills theory at finite temperature  $T$ . In the Matsubara formulation, the Euclidean time direction is topologically  $\mathbb{S}^1$  with periodicity  $\beta = 2\pi/T$ . In general, the self-dual Yang-Mills potential  $A_\mu(\mathbf{x}, t)$  on  $\mathbf{R}^3 \times \mathbb{S}^1$  obeys the quasi-periodicity condition:  $A_\mu(\mathbf{x}, t + \beta) = \omega^{-1} A_\mu(\mathbf{x}, t) \omega$  where  $\omega$  is an element of the  $\text{SU}(2)$  gauge group. By making the large gauge transformation  $A_\mu(\mathbf{x}, t) \rightarrow A_\mu^\Omega(\mathbf{x}, t) = A_\mu(\mathbf{x}, t) + i\Omega(\mathbf{x}, t) \partial_\mu \Omega^{-1}(\mathbf{x}, t)$ ,  $\Omega = \omega^{t/\beta}$ , we can bring the gauge potential strictly periodic. Semiclassically, only those gauge field configurations with the *same* gauge orientation at infinity contribute to the functional integral. Such periodic instantons, also known as calorons [12], can be found from multi-instanton solution of ‘t Hooft type, in which gauge orientation of constituent instantons is identical. We arrange the constituent instantons on the same spatial location, same size, but arrayed along the temporal direction with separation  $\beta$ . Summing over the infinite constituent instantons,

the caloron solution is given by  $A_m^a = -\eta_{mn}^a \partial_n \log \Pi(x : Z)$  where the prepotential  $\Pi$  is given by

$$\Pi(x : Z) = 1 + \frac{\rho^2 T^2}{rT} \frac{\sinh rT}{2(\cosh rT - \cos tT)} \quad (2.6)$$

on base manifold  $B = \mathbb{R}_3 \times \mathbb{S}_1$ . For later convenience, we also abbreviated  $x = (\mathbf{x}, t)$  and  $Z = (\rho, \mathbf{a}, a_0)$  and denoted  $r \equiv |\mathbf{x} - \mathbf{a}|$ ,  $t = (t - a_0)$ . Notice that the instanton size moduli  $\rho$  refers to that of the constituents, viz. of the zero temperature instantons. The temperature  $T$  is related to the Euclidean time periodicity  $\beta$  as  $\beta = 2\pi/T$ . The prepotential  $\Pi$  is manifestly nonsingular, since the denominator never vanishes, except the measure-zero configuration at the instanton center. In this case, even though the prepotential diverges, integration over  $\mathbf{R}^4$  renders it finite.

Anatomy of the caloron and its physical implications were studied thoroughly [13]. At large distance,  $|\mathbf{x}| \gg \beta$ , the prepotential is expandable as  $\Pi(x : Z) \sim 1 + \rho^2 T/2r + \mathcal{O}(e^{-rT/2\pi})$ , so the temporal and the spatial components of the gauge potential become

$$A_0^a \sim -\frac{x^a}{r^2} \left(1 + \frac{2r}{\rho^2 T}\right)^{-1} + \dots \quad \text{and} \quad A_i^a \sim \epsilon_{ij}^a \frac{x^j}{r^2} \left(1 + \frac{2r}{\rho^2 T}\right)^{-1} + \dots \quad (2.7)$$

It then follows that the color electric and magnetic fields decay as  $r^{-1}$ , and hence exhibit characteristics of magnetic monopoles. Indeed, magnetic monopole is interpretable as an infinite array of Yang-Mills instantons along the Euclidean time direction.

At short distance,  $|\mathbf{x}| \ll \beta$ , the prepotential is expandable as  $\Pi(x : Z) \sim (1 + \rho^2 T^2/12) + \rho^2/|x|^2 + \rho^2 \mathcal{O}(|x|^2/\beta)$ . Consequently, the gauge potential asymptotes to

$$A_m^a \sim \eta_{mn}^a \frac{\rho_T^2}{|x|^2} \frac{x^n}{(|x|^2 + \rho_T^2)} + \dots \quad (2.8)$$

where  $\rho_T$  is related to  $\rho$  as

$$\frac{1}{\rho_T^2} = \frac{1}{\rho^2} + \frac{T^2}{12}. \quad (2.9)$$

Notice that the gauge potential takes precisely the same form as the zero temperature Yang-Mills instanton except the crucial difference that the size moduli ought to be identified with  $\rho_T$ , not  $\rho$  itself. It is important to recall that the asymptotic expansion is valid for *any* value of  $\rho$ . It then follows that, as interpreted in terms of zero temperature Yang-Mills instanton, the caloron size moduli is limited over the range:

$$0 \leq \rho \leq \infty \quad \longrightarrow \quad 0 \leq \rho_T \leq \frac{2\sqrt{3}}{T}. \quad (2.10)$$

The fact that the asymptotic caloron behaves as the zero temperature instanton implies that the caloron ought to exhibit conformal symmetry asymptotically near  $\rho_T \sim 0$ . Beyond the leading semiclassical approximation, generically there will be corrections arising from logarithmically running coupling constants. For conformally invariant  $\mathcal{N} = 4$  Yang-Mills theory, such effects are absent, rendering the conformal symmetry at  $\rho_T \sim 0$  exact.

The result, eq. (2.10) is elementary but leads to remarkable consequences. We expect that, as measured in scales of the zero temperature instanton, the caloron size does not grow forever (which was so at zero temperature) as  $\rho \rightarrow \infty$ . Rather, caloron size saturates at a finite size set by the temperature  $T$ . It suggests us to interpret the maximum caloron size as the emergent horizon of the black hole in the background of the information geometry space  $\mathbf{H}_5$  in (2.3).

### 3. Analysis and results

Having understood the caloron configuration, we would like to understand the geometry of caloron's moduli space. As for the zero temperature instantons, we shall adopt Fisher-Rao's information metric (2.2) as the definition of the geometry. In terms of the prepotential  $\Pi(x : Z)$ , the caloron action density is expressible by  $\mathcal{L}_{\text{YM}} = \frac{1}{2} \square_{\mathbb{R}_4} (\partial_m \log \Pi)^2 = -\square_{\mathbb{R}_4}^2 \log \Pi$ . We anticipate the information metric for the caloron of fixed topological charge takes the form

$$ds_{\text{caloron}}^2 = G_{tt}[u]dt^2 + G_{uu}[u]du^2 + G_{ii}[u]d\mathbf{x}^2. \tag{3.1}$$

Here,  $u = u(\rho)$  refers to a functional relation between the size moduli  $\rho$  and the holographic coordinate  $u$  to be determined. By translational symmetry along  $\mathbb{R}^3$  and along  $\mathbb{S}^1$ , the metric components are functions of  $\rho$  only. Analytic computations of these metric component were not available, so we extracted them from numerical computation on MATHEMATICA.

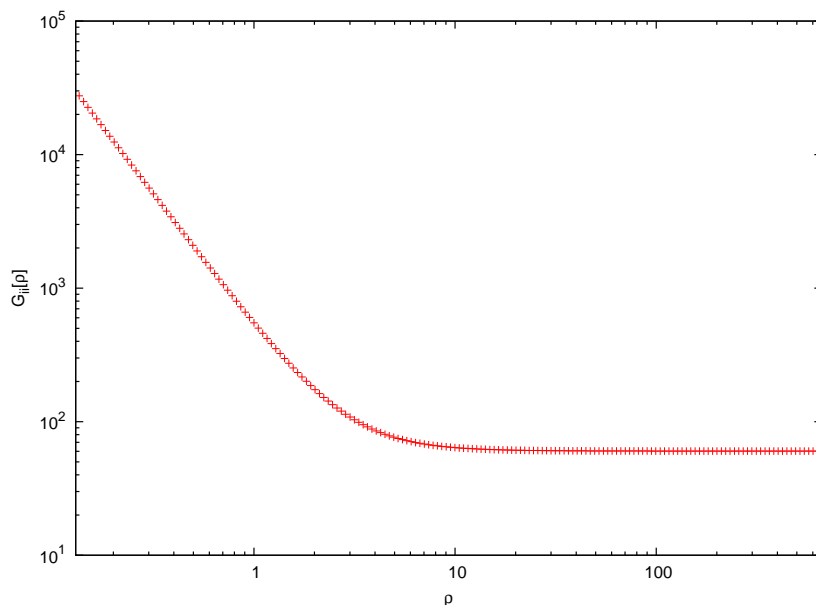
We propose to fix function  $u(\rho)$  by comparing the information geometry with the Euclidean anti-de Sitter Schwarzschild black hole. In terms of Poincaré coordinates, the metric is given by

$$ds^2 = \frac{1}{u^2} \left( 1 - \frac{u^4}{u_0^4} \right) dt^2 + \left( 1 - \frac{u^4}{u_0^4} \right)^{-1} \frac{du^2}{u^2} + \frac{1}{u^2} d\mathbf{x}^2. \tag{3.2}$$

The background has a topology of  $\mathbb{R}_2 \times \mathbb{R}_3$ , the latter subspace referring to the horizon. In Euclidean signature, the geometry is complete for  $u \geq u_0$ , and the regularity of the  $\mathbb{R}_2$  part at  $u = u_0$  relates the temperature and the mass of the black hole. In this Poincaré parametrization, the coordinate  $u$  is the radial coordinate and ranges over  $u_0 \leq u < \infty$  for Euclidean signature (compared to the range  $0 \leq u < \infty$  for Lorentzian signature). At strong 't Hooft coupling regime, leading corrections were computed in [2], and indicated that the horizon and the singularity at  $u = 0$  persist.

- $G_{ij}$ :

We begin with the metric component  $G_{ij}$ , since, according to eq. (3.2), we expect that this metric component is not deformed by the Schwarzschild harmonic function, viz. takes the same functional form  $1/\rho^2$  as the Euclidean anti-de Sitter space  $\mathbf{H}_5$ . In the previous section, we noted from asymptotic behavior that the caloron behaves the same as zero temperature Yang-Mills instanton for small size  $\rho \sim 0$ , but is gradually deformed for larger size. We anticipated that, when measured in terms of *zero temperature instanton size*



**Figure 1:** Log-log plot of the information metric  $G_{ij}$  as a function of  $\rho$ . The numerical data points are cross-marked. The Yang-Mills gas temperature is set to  $T = 1$ . Across  $\rho = 1/T$ , the metric component  $G_{ij}$  behaves very differently.

variable  $\rho_T$  in (2.9), the instanton size is bounded above at a finite size set by the Yang-Mills temperature  $T$ . Certainly, this is not the behavior one finds from the  $L^2$ -metric and ADHM calculus. So, we compared the relation (2.9) with the information metric component  $G_{ij}$ . Our computation yields

$$G_{ij}[\rho] = 506 \left( \frac{1}{\rho^2} + 0.119 \right) \delta_{ij} . \tag{3.3}$$

Remarkably, the result displays the same feature as (2.9):  $G_{ij}$  approaches to a *constant* value as  $\rho \rightarrow \infty$ ! Thus, we propose to set the functional relation  $u(\rho)$  as

$$\frac{1}{u^2} := \left( \frac{1}{\rho^2} + \frac{1}{u_o^2} \right) \quad \text{where} \quad \frac{1}{u_o^2} \sim 0.119 T^2, \tag{3.4}$$

and identify  $u$ , not  $\rho$ , with the radial variable in the metric (3.2). Stated differently, (3.4) defines the map between Yang-Mills theory scale (instanton size) and the emergent geometry scale (holographic distance). We thus have

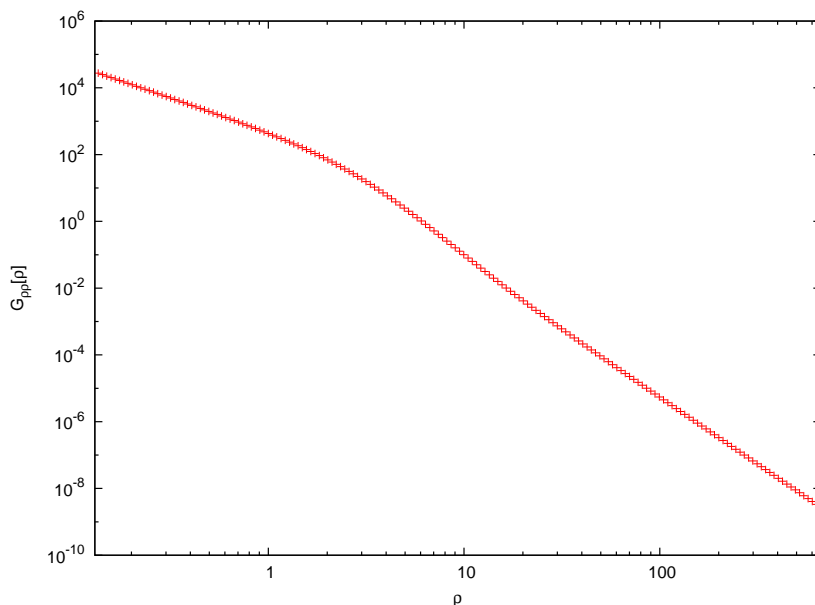
$$G_{ij}[u] = 506 \frac{\delta_{ij}}{u^2}. \tag{3.5}$$

Since the radial variable  $u$  extends only up to  $u_o$ , we identify  $u_o \sim 2.9/T$  as the "horizon". Notice that this agrees well with  $2\sqrt{3}/T$  in (2.10). Accordingly, we interpret the Yang-Mills gas temperature  $T$  as the Hawking temperature.

- $G_{uu}$ :

Next, consider the metric component  $G_{uu}[u]$ . From the anti-de Sitter Schwarzschild metric,





**Figure 2:** Log-log plot of the information metric  $G_{\rho\rho}$ . At  $\rho \ll 1/T$ , it scales as  $1/\rho^2$ . At  $\rho \gg 1/T$ , it scales as  $1/\rho^4$ .

we anticipate that  $G_{uu}$  diverges at the "horizon"  $u_0$ . From Yang-Mills theory side, we will first compute the metric component  $G_{\rho\rho}$  and then change the variable according to (3.4), convert it to  $G_{uu}[u] = G_{\rho\rho}(\rho) \left(\frac{d\rho}{du}\right)^2$ . Numerically, we obtained that  $G_{\rho\rho}$  interpolates between  $\rho^{-2}$  and  $\rho^{-4}$  behaviors as  $\rho$  ranges from 0 to  $\infty$ :

$$G_{\rho\rho}(\rho) = 502 \left( \frac{1}{\rho^2 + 0.91 \rho^4} \right). \tag{3.6}$$

The result is plotted in figure 2.

Changing the variables to  $u$ , we find that

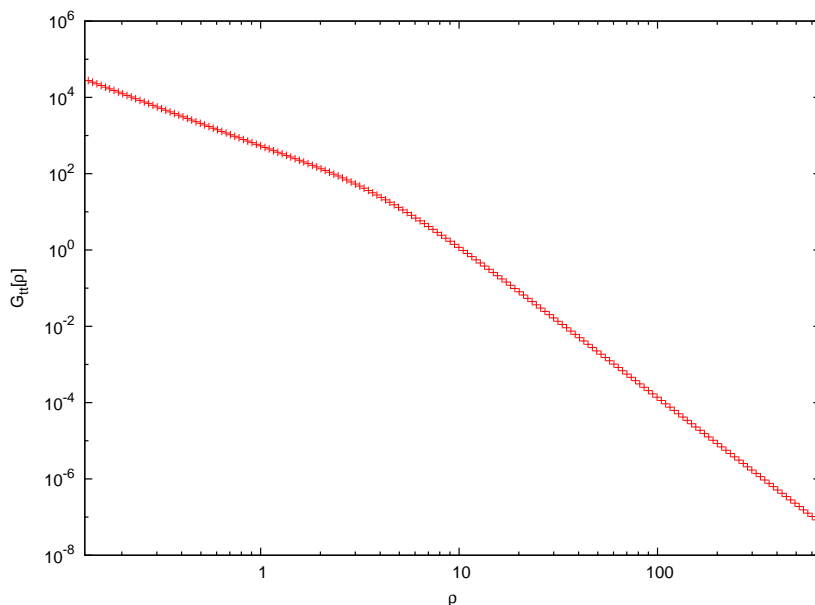
$$G_{uu}[u] = 502 \frac{1}{u^2} \left(1 - \frac{u^4}{u_0^4}\right)^{-1} R(u) \quad \text{where} \quad R(u) \sim \frac{1 + u^2/u_0^2}{1 + 6.70 u^2/u_0^2}. \tag{3.7}$$

Remarkably, the metric has a simple pole at  $u = u_0$ , which is precisely the location of the "horizon". Notice that the location of the pole originates from the change of the variables, explaining why it is invariably the same as the location of the horizon. Compared to the metric (3.2), we thus see that the metric component is now modified by the function  $R(u)$ . Since these two results are totally opposite regime of the coupling parameters, we conclude that  $R(u)$  summarizes strong worldsheet and string coupling effects.

•  $G_{tt}$ :

Lastly, we extract  $G_{tt}$  component of the information metric. From the anti-de Sitter Schwarzschild black hole metric, we expect this component exhibits a single zero at the horizon. From numerical computation, we have found that the best fit is given by

$$G_{tt}(\rho) = 507 \left( \frac{1}{\rho^2 + 0.037 \rho^4} \right). \tag{3.8}$$



**Figure 3:** Log-log plot of the information metric  $G_{tt}$ . At  $\rho \ll 1/T$ , it scales as  $1/\rho^2$ . At  $\rho \gg 1/T$ , it scales as  $1/\rho^4$ . It behaves similar to  $G_{\rho\rho}$ , but the numerical coefficients for each slope are different.

Again, changing the  $\rho$  variable into  $u$  via (3.4), we get

$$G_{tt}[u] \sim 507 \frac{1}{u^2} \left(1 - \frac{u^4}{u_0^4}\right) T(u) \quad \text{where} \quad T(u) = \frac{1 - u^2/u_0^2}{1 - 0.69 u^2/u_0^2} \left(1 + \frac{u^2}{u_0^2}\right)^{-1}. \quad (3.9)$$

We see that the metric vanishes at the "horizon",  $u \rightarrow u_0$ ! The metric actually exhibits *double* zero at the horizon. It would be interesting to see if the double zero resolves into a single zero once numerical accuracy and fitting function with higher precision are performed, but we relegate the study for future work. It also exhibits a simple pole, but it is hidden inside the horizon.

Summarizing, the Euclidean metric of the instanton moduli space takes the form

$$ds^2 \sim 500 \left[ \frac{1}{u^2} \left(1 - \frac{u^4}{u_0^4}\right) T(u) dt^2 + \left(1 - \frac{u^4}{u_0^4}\right)^{-1} R(u) \frac{du^2}{u^2} + \frac{1}{u^2} d\mathbf{x}^2 \right], \quad (3.10)$$

where  $T(u), R(u)$  given in (3.8) and (3.7) represent effects due to string worldsheet and quantum corrections.

Lorentzian black hole geometry is obtainable by Wick rotating the Euclidean metric (3.10). The Wick rotation is a well-defined notion in gauge theory, and this must be hold as well for the instanton moduli space. It then follows that the Lorentzian black hole clearly exhibits  $u = \infty$  singularity as in the case for the Schwarzschild black hole. We believe these features bear important implications to exploring inside the black hole via Maldacena's AdS/CFT [14]. In our numerical computation, there appears other poles and zeros in the metric components, all located at finite  $u$  and inside the horizon. However, further precision study is imperative before drawing physical significance of them.

Any extra structures present to the moduli space geometry bear significant implications to our understanding of string theory since they represent effects arising from unsuppressed stringy and quantum fluctuations. Another interesting direction is to reconstruct the minimal surface of Wilson loop [15]-[19] directly from weakly coupled gauge theory. We leave them for future study.

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